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Using Modified Bessel's Function to solve Heat Equation

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Abstract

In this work, a heat equation was solved by Applying boundary condition without time dependence by using a Laplace operator and Modified Bessel's function. considering a problem of heat equation and the solution of this problem implement in computer programming.

From the results we noted that, the solutions of heat equation with initial conditions according to the results obtained that dependent on M.

الملخص

في هذا العمل قمنا بإيجاد الحل لمعادلة الحرارة باستخدام دالة بيسل المعدلة و معامل لابلاس و قد تم الحصول على نتائج لهذه المعادلة باستخدام برنامج ماتلاب، ووجدنا انه عند تغيير M حيث $M=\sqrt{(2h/tk)}$ تتغير النتائج المتحصل عليها .

Keywords: Bessel function, Laplace operator, partial differential equations, Modified Bessel's function and heat equation.

1. Introduction

Bessel's and modified Bessel's equation can be used to find solution of Laplace's equation [1] which is can solved by Laplace's transform [2] and in [3] using Laplace transformation for solving heat conduction equation.

The following are definition and basic properties of modified Bessel's function.

The modified Bessel's function $I_v(x)$ and $k_v(x)$ are the solutions of the modified Bessel's equation. Figure (1). illustrated the modified bessl's function of the second kind when $v = 0,1,2,3,4$.

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + ((ix)^2 - v^2)y = 0.$$

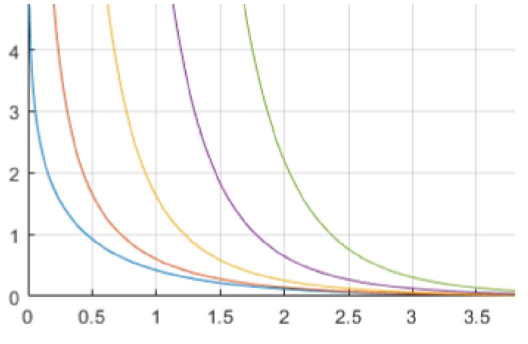

Fig.1. Modified Bessel's function

Equivalently

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + v^2)y = 0.$$

The solution of the modified Bessel equation can be written as

$$y(x) = c_1 I_v(x) + c_2 k_v(x).$$

Here $I_v(x)$, $k_v(x)$ are called the modified Bessel function of the first kind and the second kind respectively of order v (you can see [4]) where

$$I_v = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{v+2n}}{n!\,\Gamma(v+n+1)}$$

and

$$k_v = \frac{\pi[I_{-v}(x) - I_v]}{2\sin n\pi} \qquad , n = 0,1,2,\dots$$

Where the Gamma function (you can see [4],[5]) is defined by [3],[4]

$$\Gamma(p) = \int_0^{\infty} e^{-x} x^{p-1}\, dx \quad , for \quad p \neq 0, -1, -2, \dots$$

Which-has the following properties

1. $\Gamma(1) = 1.$
2. $\Gamma(p+1) = p\Gamma(p).$
3. $\Gamma(p+1) = p!\ \ for\ positive\ integral\ p\ .$

In this paper, we use a modified Bessel's functions to solve a heat equation and how these functions can be applied to finding the vibration.

2. Preliminaries

Throughout this paper, we consider the eigenvalue for the Laplacian on a bounded domain Ω, λ called an eigenvalue of the laplacian and the $v \epsilon \Omega$ function called the eigen function satisfied

$$\begin{cases} \Delta v + \lambda_n = 0 & in\ \Omega, \\ v = 0 & on\ \partial\Omega. \end{cases}$$

Definition 2.2 [2]

The solution of the Laplace equation

$$\Delta v = \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2},$$

is

$$v = \sum_{n} \sum_{m} f_{n,m} e^{in\theta} \cos\left(\frac{\pi m}{2a} r\right) \left(\frac{I_n(k_m z)}{I_n(k_m R)}\right),$$

Where $f_{n,m}$ given by

$$f_{n,m} = \frac{1}{2\pi a} \int_0^{2\pi} d\theta \int_{-a}^{a} f(\theta, r) e^{in\theta} \cos\left(\frac{\pi m}{2a}\right) dr.$$

Which satisfying the equation

$$\frac{1}{r} \frac{\partial R}{\partial r} - \left({k_m}^2 + \frac{n^2}{z^2}\right) R = 0$$

Where ${k_m}^2 = \frac{\pi m}{2a}$ $and\ R = \frac{I_n(k_m z)}{I_n(k_m)}$.

3. Solving a heat equation by using a modified Bessel's function

We study the following problem

$$\begin{cases} -\Delta v = \lambda v. \\ v(x, 0) = \varphi(x), \end{cases}$$

we solve a version of the problem without time dependence and we will use the polar coordinates.

we use the eigenvalue problem

$$\begin{cases} -\Delta v = \lambda v & (1) \\ v = 0 & (2) \end{cases}$$

Start with

$$\Delta v = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2},$$

i.e.

$$\Delta v = \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2},$$

from $v(r,\theta) = v(r)T(\theta)$,

$$\Delta v = v''(r)T(\theta) + \frac{1}{r}v'(r)T(\theta) + \frac{1}{r^2}v(r)T''(\theta),$$

by applying (1), we find

$$\frac{v''}{v} + \frac{1}{r}\frac{v'}{v} - \frac{1}{r^2}\frac{T''(\theta)}{T(\theta)} = \lambda,$$

This is

$$r^2\frac{v''}{v} + r\frac{v'}{v} - \frac{T''(\theta)}{T(\theta)} = \lambda r^2$$

It assumes that $\frac{T''(\theta)}{T(\theta)} = -m^2$, then

$$v'' + \frac{1}{r}v' - \left(\lambda + \frac{m^2}{r^2}\right)v = 0$$

Then we get

$$r^2 v'' + r v' - r^2\left(\lambda + \frac{m^2}{r^2}\right)v = 0$$

Then

$$r^2 v'' + r v' - (r^2\lambda + m^2)v = 0$$

The last equation is a modified Bessel's equation, if we assume that $\mu = \sqrt{\lambda} r$, then

$$\lambda v_{\mu\mu} + \lambda \frac{1}{\mu} v_{\mu} - \lambda \left(1 + \frac{m^2}{\mu^2}\right) v = 0$$

$$v_{\mu\mu} + \frac{1}{\mu} v_{\mu} - \left(1 + \frac{m^2}{\mu^2}\right) v = 0$$

This is modified differential equation of order m.

considered one particular example in heat transfer.

The energy balance of heat differential volume can be stated as

$$-kA \frac{dT}{dr}\bigg|_{r} = -kA \frac{dT}{dr}\bigg|_{r+dr} + hA_c (T - T_\infty)$$

Where A difined by $2\pi rt$, $A_c = 2(2\pi r dr)$,
$k = thermal\ conductivity$,
$h = converctive\ heat\ transfer\ cofficient$, r is the radial coordinate and T_∞ is the air temperature

Substituting in the area parameters and rearranging gives

$$\frac{r \frac{dT}{dr}\Big|_{r+dr} - r \frac{dT}{dr}\Big|_{r}}{dr} - \frac{2hr}{tk}(T - T_\infty) = 0.$$

When $dr \to 0$, this relation becomes

$$\frac{d}{dr}\left(r \frac{dT}{dr}\right) - \frac{2hr}{tk}(T - T_\infty) = 0.$$

Whish can be written as

$$r^2 \frac{d^2T}{dr^2} + r \frac{dT}{dr} - \frac{2hr^2}{tk}(T - T_\infty) = 0.$$

Which is modified Bessel differential equation of order 0.

The last equation can be simplified and put in form, first we introduce the shifted temperature and let $\tilde{r} = \sqrt{2h/tkr}$ and let $\theta = T - T_\infty$

The last relation becomes

$$\tilde{r}^2 \frac{d^2\theta}{d\tilde{r}^2} + r\frac{d\theta}{d\tilde{r}} - \tilde{r}^2\theta = 0.$$

Which is the standard form of the modified Bessel of order zero.

Thus, the solution is given by

$$\theta = c_1 I_0(Mr) + c_2 K_0(Mr).$$

Where $M = \sqrt{2h/tk}$ and c_1, c_2 are arbitrary constants to be determine by two boundary conditions.

Consider the specific problem with temperature boundary conditions

$$\theta(r_1) = 100, \theta(r_0) = 10.$$

Using the general solution into these two conditions gives

$$c_1 I_0(Mr_1) + c_2 K_0(Mr_1) = 100.$$

$$c_1 I_0(Mr_0) + c_2 K_0(Mr_0) = 10.$$

These boundary condition equations represent two equations for the two constants
c_1 and c_2. .

The equation system can be easily solved by using Cramer's Rule (see Kreyzig, p298) to be

$$c_1 = \frac{100K_0(Mr_0) - 10K_0(Mr_1)}{I_0(Mr_1)K_0(Mr_0) - I_0(Mr_0)K_0(Mr_1)}$$

$$c_2 = \frac{10I_0(Mr_1) - 100I_0(Mr_1)}{I_0(Mr_1)K_0(Mr_0) - I_0(Mr_0)K_0(Mr_1)}$$

Thus, the solution for temperature is completed.

Now we want to see the results graphically by using MATLAB to evaluate and plot the temperature distribution for the particular case with $T_\infty = 50, r_1 = 1$ and $r_0 = 5$ and three different values of M.

The results are given in the figure below and the associated MATLAB code is listed in the text box.

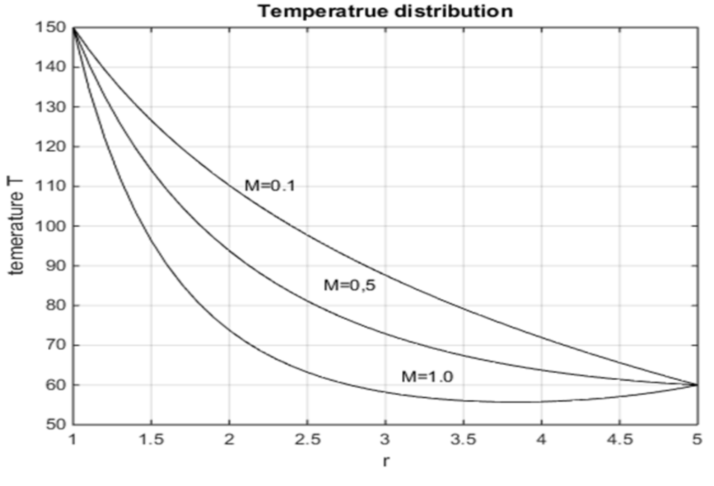

Figure 2, Temperature distribution

MATHLAB example

```
clc;clf;clear all
r1=1;r0=5;Tin=50;
r=[r1:0.1:r0];
for M=[0.1,0.5,1]
    I1=besseli(M,M*r1);I0=besseli(M,M*r0);
    k1=besselk(M,M*r1);k0=besselk(M,M*r0);
    c1=(100*k0-10*k1)/(I1*k0-I0*k1);
    c2=(10*I1-100*I0)/(I1*k0-I0*k1);

th=Tin+c1*besseli(M,M*r)+c2*besselk(M,M*r);
    plot(r,th,'k','linewidth',0.2)
    xlabel('r'),ylabel('temerature T')
    title('Temperatrue distribution')
    grid on;hold on;
end
text(2.1,110,'M=0.1')
text(2.6,85,'M=0,5')
text(3.1,62,'M=1.0')
```

4. Conclusion

In this paper, the solutions of heat equation with initial conditions according to the results obtained that dependent on M, where $M = \sqrt{2h/tk}$ we take three different values on M as we see in the figure 2 when $M = 0.1$,0.5 , 1 and every time we get different solution to the heat equation, we used MATLAB computer programming to draw figure 1,2.

5. References

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