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Topological Classification of Sets in Approximation Spaces

Entesar Al Amin

Department of Mathematics, Faculty of Science, University of Zawia,

entsaralamin77@gmail.com, an.sulayman@zu.edu.ly

التصنيف الطوبولوجي للمجموعات في الفضاءات التقريبية

انتصا ر الأمين محمد عمر-قسم الرياضيات – كلية العلوم – جامعة الزاوية – ليبيا

الملخص العربي:

يمكن النظر الى هذا البحث على انه تصنيف للمجموعات في بعض الفضاءات التقريبية باستخدام البنية الطوبولوجية العامة. نعتمد في طريقتنا على طوبولوجيا عامة تم انشاؤها بواسطة علاقة ثنائية لتوليد فضاء جديد يسمى الفضاء التقريبي المسبق والذي يعتمد على مجموعات ما قبل الفتح، وهي طريقة حديثة للاستدلال حول البيانات، وقد حقق هذا الفضاء الجديد نجاحا عظيما في العديد من المجالات في تطبيقات الحياة العملية، كما قمنا بتوضيح أنواع مختلفة من المجموعات المعرفة، بالإضافة الى ذلك فإننا ميزنا هذا الفضاء التقريبي باستخدام مناطق جديدة مختلفة معرفة من خلال تمثيل مفهوم معين مؤكد، وتوصلنا الى ان هذا الفضاء أكثر دقة وتعميم لفضاء بولاك وذلك لأننا استطعنا تعريف وتحديد بعض العناصر ومناطق الحدود والتي كانت غامضة في فضاء بولاك. هذه التقنية المقدمة مفيدة لان مفاهيم وخصائص الطوبولوجيا المولدة يتم تطبيقها على نظرية المجموعات التقريبية وهذا يفتح المجال لتطبيقات طوبولوجية أكثر في هذا السياق. إضافة الى ذلك قمنا بتقديم العديد من الخصائص مدعمين ذلك بالبراهين والكثير من الأمثلة التوضيحية.

Abstract:

This paper can be viewed as a classification of sets in some approximation spaces using general topological structure. Our approach depends on a general topology generated by binary relation to generate new approximation space called Pre-approximation space, which depends of pre-open sets and it is a recent approach for reasoning about data, this space has achieved a great success in many fields of real life applications. The different types of definability of sets are explained. In addition, we characterize the approximation spaces with different regions by representing certain concept of interest .The introduced technique is useful because the concepts and the properties of the generated topology are applied on rough set theory and this open the way for more topological applications in rough context. Several properties and examples are provided.

Keywords: Lower approximation, Upper approximation, Rough sets, Definable sets, Pre-rough sets.

1.Introduction:

In a classical set theory, a set is uniquely determined by its elements. In other words, it means that every element must be uniquely classified as belonging to the set or not. That is to say, the notion of a set is crisp (precise). On otherwise precise reasoning would be impossible. Rough set theory proposed by pawlak [9, 13] is an extension of set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. Many researchers are interested to generalize this theory in many fields of applications [1, 2, 10, 11 and 12]. The topology induced by binary relations is used to generalize the basic rough set concepts. The suggested topological Structure open up the way for classification of sets and applying rich amount of topological facts and methods in the process of granular Computing.

A new space of sets with a general relation which is introduced and studied by Entesar [3], this new space is approximation space through a pre-open sets which is studying by Abd-EL Monsef [5]. The study of pre-approximation space is deeper than the study of Pawlak space, because some elements in this space will be defined well, while these points was undefinable in Pawlak space. This paper is organized as follows: In Section 2, we give basic concepts of Pawlak space. The concept of rough sets in topological approximation spaces will be discussed in Section 3. Section 4 is devoted to study the definability in general approximation spaces. The main goal of section 5 is to introduce classification of sets in pre-approximation spaces. The paper's conclusion is given in Section 6.

2. Definable sets in Pawlak space:

Motivation for rough set theory has come from the need to represent sub sets of a universe in terms of equivalence classes of partition of the universe. The partition characterizes a topological space; called approximation space (U, R), where U is a set called the universe and R is an equivalence relation [6, 9]. The equivalence classes of R are also known as the granules, elementary sets or blocks, we will use

$R_x = [x]_R$ to denote the equivalence classes containing x.

In the approximation Space, we consider two operators, the upper and the lower approximations of subset $X \subseteq U$ as follows:

$\overline{R}(X) = \{x \in U: R_x \cap X \neq \emptyset\}$,

$\underline{R}(X) = \{x \in \mathrm{U}: R_x \subseteq X\}$.

Boundary, positive and negative regions are also defined as follows:

$B_R(X) = \overline{R}(X) - \underline{R}(X)$, $POS_R(X) = \underline{R}(X)$,
$\text{NE}G_R(X) = \text{U} - \overline{R}(X)$.

Definition 2.1 Let (U, R) be an approximation space, $X \subseteq U$. Then:

1-X is a rough set with respect to R if and only if $\underline{R}(X) \neq \overline{R}(X) \neq X$.

2-X is R- definable (exact) set with respect to R if and only if $\underline{R}(X) = \overline{R}(X) = X$.

From this definition, we can conclude immediate the following proposition.

Proposition 2.1 Let (U, R) be an approximation space, $X \subseteq U$. Then:

1- X is a rough set with respect to R if and only if $B_R(X) \neq \emptyset$

2- X is R - definable (exact) set with respect to R if and only if $B_R(X) = \emptyset$

Corollary 2.1 Let (U, R) be an approximation space, $X \subseteq U$. Then:

1- $\underline{R}(X)$ and $\overline{R}(X)$ are definable (exact) sets.

2- $[x]_R$ is definable (exact) set.

The degree of completeness can be also characterized by the accuracy measure in which $|x|$ represents the cardinality of X as follows: $\propto_R (X) = \frac{|\underline{R}(x)|}{|\overline{R}(X)|}$, $X \neq \emptyset$. 0bviously (o $\leq \propto_R (X) \leq 1$).

Proposition 2.2 Let (U, R) be an approximation space, $X \subseteq U$. Then:

1- X is R - definable (exact) set with respect to R if $\propto_R (X) = 1$.

2- X is rough set if (o $\leq \propto_R (X) < 1$).

Proof: It is obvious.

Rough set can be also introduced using rough membership functions [9], Namely $\eta_X^R(x) = \frac{|[x]_R \cap X|}{|[x]_R|}$, $x \in U$.

Different values defines Boundary ($0 < \eta_X^R(x) < 1$), Positive
($\eta_X^R(x)=1$) and Negative ($\eta_X^R(x) = 0$) regions. The membership function is a kind of conditional probability and its value can be interpreted as a degree of certainty to which x belongs to X.

3. Rough sets in topological spaces:

The reference space in rough set theory is the approximation space whose topology is generated by the equivalence classes of R. This topology belongs to a special class known by clopen topology. In which every open set is closed. Clopen topology is called quasi-discrete topology. We will use topology, in other words, the "approximation space" is a topological space.

Definition 3.1 A topological space [4] is a pair (U, τ) consisting of a set U and a family τ of subsets of U satisfying the following Conditions:

1) $U \in \tau$ and $\emptyset \in \tau$.
2) τ is closed under arbitrary union.
3) τ is closed under finite intersection.

The elements of τ are called points of the space and the subsets of U belonging to τ are called open sets in the space and the complement of these sets are called close sets in the space, the family τ of open subsets of U is called a topology for U.

Definition 3.2 A family $\beta \subseteq \tau$ is called a base for (U, τ) if every non-empty open subset of U can be represented as a union of subsets of β . Clearly, a topological space can have many based. A family S $\subseteq \beta$ is a sub base for (U, τ) if and only if the family of all finite intersections is a base for (U, τ).

Definition 3.3 The interior of a subset $A \subseteq U$:
$A^o = U\{G \subseteq U; G \subseteq$ A and $G$is open $\}$. Evidently, A^o is the union of all open subsets of U which containing in A . Note that A is open iff $A = A^o$.

Definition 3.4 $\overline{A} = \cap \{F \subseteq U; A \subseteq F \ and \ F \ is \ closed\}$ is called the closure of subset $A \subseteq U$. Evidently, $\overline{A}$ is the smallest closed subset of U which contains A . Note that A is closed iff $\overline{A} = A$. And $B(A) = \overline{A} - A^o$ is called the boundary of a subset $A \subseteq U$.

From definition 2.1 and definition 3.3, we have the following proposition.

Proposition 3.1 Let (U, τ) be a topological space $X \subseteq U$, let $\overline{X}, X^o$ and $B(X)$ be closure, interior and boundary points respectively. Then:

(i) X is exact if $\overline{X} = X^o$. Its clear that X is exact iff $B(X) = \emptyset$.

(ii) X is rough if $\overline{X} \neq X^o$and so $B(X) \neq \emptyset$.

Example 3.1 Let $U = \{a, b, c, d\}$,
τ =$\{U, \emptyset$,{a,b,c},{b,c,d},{d},{b,c}$\}$. Then we get to $X_1 = \{a, b, c\}$, $X_2 = \{d\}$ are exact sets. And X is rough set, $\forall X \in P(U)$,
$X \neq X_1, X \neq X_2$.Where $P(U)$ is the power set of U.

Proposition 3.2 If τ is the quasi-discrete topology, then Proposition 3.1 concedes with Pawlak space.

Proof: In quasi-discrete space, every open set is closed and thus the base of τ is a partition that yields an equivalence relation, which is basic tools in Pawlak space.

Remark 3.1 If τ is a general space, not quasi-discrete, then

$\overline{X}$=X $\longrightarrow$ X^o=X is not generally true. The following example ensures this fact.

Example 3.2 Let U = {1,2,3,4,5}, X = {1,2},

τ={ U, $\emptyset$,{1,2},{2,3,4},{5},{2},{2,5},{1,2,3,4},{1,2,5},{2,3,4,5}}

We have $\overline{X}$ = {1,2,3,4}, and X^o = {1,2}, then $X = X^o$, $\overline{X} \neq X$.

According to remark 3.1 and example 3.2, our approach is a generalization of Pawlak approach.

Remark 3.2 In Pawlak space, a subset $X \subseteq U$ has two possibilities; rough or exact. For general topological space, $X \subseteq U$ has the following types of definability:

(i)- X is totolly definable if X is exact set $\overline{X} = X = X^o$.

(ii)- X is internally definable if $X = X^o, X \neq \overline{X}$.

(iii)- X is externally definable if $X \neq X^o, X = \overline{X}$.

(iv)- X is undefinable if $X \neq X^o, X \neq \overline{X}$.

Example 3.3 Let $U = \{1,2,3,4,5\}, X_1 = \{1,2,3,4\}, X_2 = \{1,2\}$,

$X_3 = \{1,5\}$,

$X_4 = \{1,3\}$,$\tau =$

$\{U, \emptyset, \{1,2\}, \{2,3,4\}, \{5\}, \{2\}, \{1,2,3,4\}, \{1,2,5\}, \{2,3,4,5\}, \{2,5\}\}$.

and then we get to X_1 is exact set , X_2 is internally definable set, X_3 is externally definable set , and X_4 is undefinable set .

Proposition 3.3 If A is an exact set in a topological space (U, τ) and $\tau \subseteq \acute{\tau}$, then A is exact with respect to $\acute{\tau}$.

Proof: since $B_{\acute{\tau}}(A) \subseteq B_\tau(A)$ *and* $B_\tau(A) = \emptyset$. Then $B_{\acute{\tau}}(A) = \emptyset$ and A is exact with respect to $\acute{\tau}$. In other words, if A is $\tau -$ exact then

A is $\tau - cl$open and consequently $\acute{\tau} - clopen$. Hence A is $\acute{\tau} - exact$.

It is easy to have examples for a τ \`_exact set which is not τ _exact.

Example 3.4

Let $U = \{a, b, c, d\}, \tau` = \{U, \emptyset, \{a\}, \{b\}, \{b, c, d\}, \{a, b\}\}$ and $\tau$ = {$U$, ∅, {a}, {a,b}}, where $\tau \subseteq \tau$ `. Then {a} and {b,c,d} are τ`_exact but not τ _exact.

The following proposition gives the condition for τ `_exact sets to be τ _exact sets.

Proposition 3.4 If (U, R) is a space and $\tau \subseteq \tau$ `, then each exact A in $\tau$ ` is exact in τ iff cl_{τ}A=$cl_{\tau`}$A.

Proof: If A is τ `_exact then $A \in \tau$ ` and $cl_{\tau`}$A =$A$ and $cl_{\tau}$A =A, hence $cl_{\tau}$A =$cl_{\tau`}$A.

Conversely: if cl_{τ}A =$cl_{\tau`}$A and A is $\tau$ `_exact. Then A is τ _exact.

The topology τ defines the equivalence relation $E(\tau)$ on the power set $P(U)$ given by the condition:

$$(X, Y) \in E(\tau)\ iff\ X^o = y^o and\ \overline{X} = \overline{Y}. \quad (3.1)$$

The equivalence $E(\tau)$ can be defined in an alternative way:

$$(X, Y) \in E(\tau)\ iff\ X^o = y^o and\ b(X) = b(Y). \quad (3.2)$$

Remark 3.3 Let U be a finite non-empty universal. Then the pair $(U, E(\tau))$ is Pawlak approximation space, where $E(\tau)$ is an equivalence relation on $P(U)$ and defined by (3.1) or (3.2).

4- Definability of sets in a general approximation space

Our approach depends on the concept of after and fore sets. Let the non-empty set U, then the after set (resp. the fore) set of element $x \in U$ is the class $xR = \{y \in U: xRy\}$ (resp. $Rx = \{y \in U, yRx\}$), and the pair (U, R) is called a general approximation space (GAS) where R is a binary general relation.

Definition 4.1 Let (U, R) be a GAS and $X \subseteq U$. Then X is called "after composed" (resp. after– c composed) set if X contains all after (resp. fore) sets for all elements of its i.e., $\forall x \in X, xR \subseteq X$ (resp. $Rx \subseteq X$).

The class of all after (resp. after - c composed) sets in (U, R) is defined by the class $\tau_R = \{X \subseteq U : \forall x \in X\ ;\ xR \subseteq X\}(resp.\ \tau_R^* = \{X \subseteq U : \forall x \in X\ ; Rx \subseteq X\})$.

Pawlak space can be considered as special case of general approximation space, which the class τ_R in this case is coinciding with the class of composed sets in Pawlak space.

Proposition 4.1 let (U, R) be a GAS. Then the class τ_R (resp. τ_R^*) in (U, R) forms a topology on U.

Proof: Obvious

We have the following theorem, see [4] for more details.

Theorem 4.1 Let (U, R) be a GAS. Then τ_R is the complement topology of τ_R^* and vice versa.

Definition 4.2 Let (U, R) be a GAS and $X \subseteq U$. X is called R-definable (exact) set in (U, R) if X and X^c are after composed set. Otherwise, X is called R -undefinable (rough) set. The lower (resp. the upper) approximation of any subset $X \subseteq U$ is given by:

$\underline{R}(X) = \cup\{G \in \tau_R : G \subseteq X\}\ (resp.\ \overline{R}(X) = \cap\{H \in \tau_R^* : X \subseteq H\})$.

The boundary, the internal edge of X and the external edge of X regions are also defined as follows:

The boundary set of X is given by $B(X) = \overline{R}(X) - \underline{R}(X)$.

The internal edge of X is given by $\underline{Ed}_R\ (X) = X - \underline{R}(X)$.

The external edge of X is given by $\overline{Ed}_R\ (X) = \overline{R}(X) - X$.

Remark 4.1

1-It is easy to notice that the lower $\underline{R}(X)$ (resp. the upper $\overline{R}(X)$) approximation of a subset X in GAS (U, R) is exactly the interior X^o (resp. the closure $\overline{X}$) of X in the topology τ_R.

2- It is clear that $B(X) = \underline{Ed}_R\ (X) \cup \overline{Ed}_R\ (X)$.
3-The best lower (resp. upper) approximation of any subset is given when the internal (resp. the external) edge of its tends to empty set, that is: if $\underline{Ed}_R\ (X) = \ \emptyset\ \left(resp. \overline{Ed}_R\ (X) = \emptyset\right)$.
4- $\underline{R}(X)(resp. \overline{R}(X))$ is the largest after (resp. smallest after-c) composed set contained in X (resp. contain X).
5- X is after (resp. after-c) composed set iff $\underline{R}(X) = X$
$(resp. \overline{R}(X) = X)$.

Remark 4.2 We can get the approximation operators as follows:
1-Get the family of after sets xR from the given relation $xR = \{y \in U: xRy\}$
2-Using the class of all after sets xR as a sub-base $S = \{xR, x \in U\}$ for the topology τ as shown in the following example.

Example 4.1 let $U = \{0,1,2,3,4,5\}, oR = 1R = \{0,1,2\}, 2R = 3R = \{2,3\}$,$4R = \{3,4\}$, $5R = \{5\}$. Then the set
$S = \left\{\{0,1,2\}, \{2,3\}, \{3,4\}, \{5\}\right\}$ is a sub base and
$\beta = \left\{\{0,1,2\}, \{2,3\}, \{3,4\}, \{5\}, \{2\}, \{3\}, \emptyset\ \right\}$ is a base for the following topology
$\tau = \{U, \emptyset, \{0,1,2\}, \{2,3\}, \{3,4\}, \{5\}, \{2\}, \{3\}, \{0,1,2,3\}, \{0,1,2,3,4\},$
$\{0,1,2,5\}, \{2,3,4\}, \{2,3,5\}, \{3,4,5\}, \{2,5\}, \{3,5\}, \{2,3,4,5\}\}$.
$Let\ X = \{0,1,2,3\}, then\ \underline{R}(X) = X^o = \{0,1,2,3\}, \overline{R}(X) = \overline{X} = \{0,1,2,3,4\}, So\ X\ is\ rough\ set.\ \ POS_R(X) = \{0,1,2,3\}, B(X) = \{4\},\ \mathrm{NEG}_R(X) = \{5\}, \underline{Ed}_R\ (X) = \emptyset\ , \overline{Ed}_R\ (X) = \{4\}$.

Definition 4.3 Let (U, R) be a GAS. Then the subset $X \subseteq U$ is said to be:
(i)Totally-definable (exact) set if $X = \underline{R}(X) = \overline{R}(X)$
$(i.e., B(X) = \emptyset)$.
(ii) Internally definable set if $X = \underline{R}(X)$, such that $\underline{Ed}_R\ (X) = \emptyset$.

(iii) Externally-definable set if $X = \overline{R}(X)$, such that $\overline{Ed}_R\ (X) = \emptyset$.

(iv) Undefinable or (rough) set if $X \neq \underline{R}(X) \neq \overline{R}(X)$ $(i.e., B(X) \neq \emptyset)$.

Remark 4.3 In the above definition $B(X) \neq \emptyset$ in cases (ii), (iii).

Example 4.2 In Example 4.1, $\underline{Ed}_R\ (X) = \emptyset$, $\overline{Ed}_R\ (X) = \{4\}$. So X is internally-definable set, let $Y = \{3,4,5\}, W = \{4,5\}$ and Z={0,1,2,4},we have Y is Totally-definable (exact) set, ,W is Externally-definable set and Z={0,1,2,4} is undefinable or (rough) set.

Lemma 4.1 Let (U, R) be a GAS and $X \subseteq U$. Then:

(i) X is internally definable set if and only if it is after composed set.

(ii) X is externally definable set if and only if it is after-c composed set.

Proof: Obvious.

Proposition 4.2 Let (U, R) be a GAS and $X \subseteq U$. Then:

(i) X is exact set if and only if X is internally and externally definable set.

(ii) X is rough set if and only if X is neither internally nor externally definable set.

Proof: By Lemma 4.1, the proof is obvious.

Remark 4.4 From the above proposition and lemma, we have:

(i) X is exact iff it is after and after-c composed set.

(ii) X is rough iff it is neither after composed nor after-c composed set.

By considering τ_R of a GAS(U, R), forms a topology on U, then we can reformulate Definition 3.1 by a topological view as follow:

Definition 4.4 Let (U, R) be a GAS with a topology τ_R on U,

$X \subseteq U$. Then:
(i) X is said to be internally (resp. externally, totally) definable set if X is open (resp. closed, clopen) set inτ_R.
(ii) X is said to be R-undefinable (rough) set if X neither open nor closed set in τ_R.
Remark 4.5 According to Definition 3.2, we have seen how the topology represents the magic box for definability of sets. Thus, the collection of open and closed sets represents the golden tools to measure exactness or the roughness of sets

5. Classification of sets in pre-approximation space

In [3] has been introduced the new concept of pre-approximation space, which based on the class of pre-open sets and deal with general binary relations. This new space based on topological operators.
We have the following definitions, see [3,5] for more details.
Definition 5.1 Let (U, τ) be a topological space. The subset $X \subseteq U$ is called pre-open if $X \subseteq int\big(cl(X)\big)$. The complement of a pre-open set is called pre-closed, the family of all pre-open sets of (U, τ) is denoted by $P_O(U, \tau)$ and the family of all pre-closed sets of (U, τ) is denoted by $P_C(U, \tau)$.
Definition 5.2 Let U be a finite nonempty universal. Then the pair (U, R_P) is called a pre-approximation space, where R_P is a general relation referes as a subbase for topology, τ and used to generates a class of pre-open sets $P_O(U, \tau)$.
Pre-rough classification
Definition 5.3 Since for any $X \subseteq U$, we have
$int(X) \subseteq p.int(X) \subseteq X \subseteq p.cl(X) \subseteq cl(X)$. (5.1)
and based on the lower (interior) and upper (closure) approximation, also pre-interior and pre-closure. The universe U se

can be divided into 12 regions with respect to any $X \subseteq U$ as follows:

(1) The internal Edg of X :
$\underline{Ed}(X) = X - int(X)$.
(2) The pre-internal Edg of X :
$Pre.\underline{Ed}(X) = X - p.int(X)$.

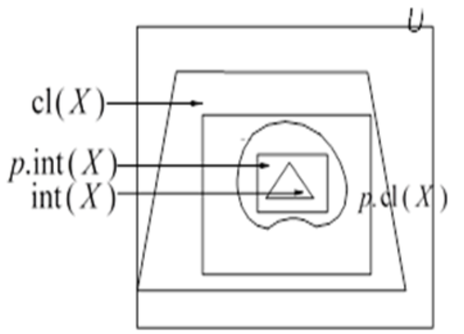

Figure (5.1)

From (1) and (2) observe that we can be written (1) as:
$\underline{\mathrm{Ed}}(X) = \mathrm{pre}.\underline{\mathrm{Ed}}(X) \cup (p.\mathrm{int}(X) - \mathrm{int}(X))$.

(3) The external Edg of X : $\overline{\mathrm{Ed}}(X) = \mathrm{cl}(X) - X$.

(4) The pre-external Edg of X: $\mathrm{pre}.\overline{\mathrm{Ed}}(X) = p.\mathrm{cl}(X) - X$.

Also, we can be written (3) as:

$\overline{\mathrm{Edg}}(X) = \mathrm{pre}.\overline{\mathrm{Edg}}(X) \cup (\mathrm{cl}(X) - p.\mathrm{cl}(X))$.

(5) The boundary of X : $B(X) = \mathrm{cl}(X) - \mathrm{int}(X)$.

(6) The pre-boundary of X: $p.B(X) = p.\mathrm{cl}(X) - p.\mathrm{int}(X)$.

(7) The exterior of X : $ex(X) = U - \mathrm{cl}(X)$.

(8) The pre-exterior of X: $p.ex(X) = U - p.\mathrm{cl}(X)$.

(9) $\mathrm{cl}(X) - p.\mathrm{int}(X)$.

(10) $p.\mathrm{cl}(X) - \mathrm{int}(X)$.

(11) $p.int(X) - \mathrm{int}(X)$.

(12) $\mathrm{cl}(X) - p.\mathrm{cl}(X)$.

Remark 5.1: As shown in Figure (5.1). The study of pre-approximation space is deeper than the study of approximation space and generalization for it. Because of the elements of the region (11) will be defined well in X, while these points was undefinable in Pawlak approximation space. In addition, the elements in (12) is not belong to X, while these elements was not well defined in Pawlek space. In our study, we replace the boundary of X (5) by Pre-boundary of X (6). Also, we extend exterior of X (7) by pre-exterior of X (8).

Proposition 5.1 For every $X \subseteq U$, we have:

(i) $p.\mathrm{cl}(X) = p.\mathrm{int}(X) \cup p.B(X)$.

(ii) $p.B(X) = p.\underline{\mathrm{Ed}}(X) \cup p.\overline{\mathrm{Ed}}(X)$.

Remark 5.2 For every subset $X \subseteq U$, we have: $p.B(X) \subset B(X)$ and $ex(X) \subset p.ex(X)$.

As an approximations space, we can define the pre-lower and pre- upper approximations (respectively) as follows:

Definition 5.4 [3] let (U, R_P) be a pre-approximation space, the pre- lower and pre-upper approximations of a non-empty subset X of U is defined as:

$$\underline{R}_p(X) = \bigcup\{\mu \in P_0(U, \tau);\ \mu \subseteq X\}$$

$$\bar{R}_P(X) = \bigcap\{N \in P_c(U, \tau),\ X \subseteq N\}$$

Its obviously, one can see the pre-lower and pre-upper approximations are pre-interior and pre-closure of any set as shown in Figure (5.2).

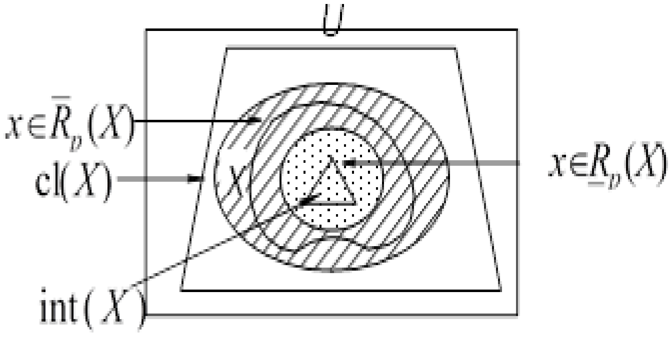

Figure (5.2)

Definition 5.5 The pre-boundary of X is denoted by $P.B(X)$ is given by $p \cdot b(X) = \bar{R}_p(X) - \underline{R}_p(X)$.

Properties of this new space has been established and studied by Entesar (see[3]).

From the relation (5.1) and definition 5.2, we have the following theorem.

Theorem 5.1 Let (U, R_P) be a pre-approximation space, $X \subseteq U$ we have: $\underline{R}(X) \subseteq \underline{R}_P(X) \subseteq X \subseteq \bar{R}_P(X) \subseteq \bar{R}(X)$.

Proof: $\underline{R}(X) = \cup \{G \in \tau_R : G \subseteq X\} \subseteq \cup \{G \in P_O(\mathrm{U}, \tau), G \subseteq X\} = \underline{R}_P(X) \subseteq X$, that is $\underline{R}(X) \subseteq \underline{R}_P(X) \subseteq X$. Also, $\bar{R}(X) = \cap \{H \in \tau_R^* : X \subseteq H\} \supseteq \cap \{H \in P_c(\mathrm{U}, \tau) : X \subseteq H\} = \bar{R}_P(X) \supseteq X$.
So $\underline{R}(X) \subseteq \underline{R}_P(X) \subseteq X \subseteq \bar{R}_P(X) \subseteq \bar{R}(X)$.

Definition 5.6 Let (U, R_P) be a pre-approximation space, and let $X \subseteq U$, then we define pre-boundary of as follows:

$p.B(X) = \bar{R}_P(X) - \underline{R}_p(X)$.

Definition 5.7 Pre-strong membership and pre-weak membership

Let (U, R_P) be a pre-approximation space and $X \subseteq U$, then there are memberships $\underline{\in}_p, \overline{\in}_p$ call them pre-strong membership and pre-weak membership respectively which defined by:

(i) $x \underline{\in}_p X$ iff $x \in \underline{R}_p(X)$, where $x \underline{\in}_p X$ means x surely belongs to X.

(ii) $x \overline{\in}_p X$ iff $x \in \overline{R}_p(X)$, where $x \overline{\in}_p X$ means x possibly belongs to X.

Remark 5.3 According to above definition, the pre-lower and pre-upper approximation can be written to:

$\underline{R}_p(X) = \{x \in X : x \underline{\in}_p X\}$ and $\overline{R}_p(X) = \{x \in X : x \overline{\in}_p X\}$.

Remark 5.4 If $x \underline{\in} X$ then $x \underline{\in}_p X$ and $x \overline{\in}_p X$ implies $x \overline{\in} X$ but the converse is not true in general as it can be seen by following example.

Example 5.1 Let $U = \{a, b, c, d\}$ and R is defined as $aR = \{a\}$, $bR = \{b, c\}$Then the topology which associated with this relation is $\tau = \{U, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. If $X = \{a, c, d\}$,we have $\underline{R}(X) = \{a\}, \overline{R}(X) = U$, $\underline{R}_p(X) = \{a, c, d\}$ and $\overline{R}_p(X) = \{a, c, d\}$.Then $c \underline{\in}_p X$ but $c \underline{\notin} X$, while, $b \overline{\in} X$ and $b \overline{\notin}_p X$.

Pre-rough sets:

Definition 5.8 Let (U, R_P) be a pre-approximation space, $X \subseteq U$. Then:

1- X is called R_p-definable (pre- exact) whenever $\overline{R}_p(X) = \underline{R}_p(X)$.

2- X is a pre- rough whenever $\overline{R}_p(X) \neq \underline{R}_p(X)$.

Remark 5.5 It is easy to notice that the pre-lower $\underline{R}_P(X)$ (resp. the pre-upper $\overline{R}_p(X)$) approximation of a subset X in pre-approximation space (U, R_P) is exactly the pre-interior $p.int(X)$ (resp. the pre-closure $p.cl(X)$) of X in the topology τ_R.

Definition 5.9 Let (U, R_p) be a pre-approximation space, $X \subseteq U$ then:

(i) X is a pre- exact if $p.\mathrm{int}(X) = p.\mathrm{cl}(X)$.

(ii) X is a pre- rough if $p.\mathrm{int}(X) \neq p.\mathrm{cl}(X)$.

Proposition 5.2 Let (U, R_p) be a pre-approximation space, $X \subseteq U$ then:

1) X is called R_p-definable iff $p.b(X) = \varnothing$

2) X is a pre- rough whenever $p.b(X) \neq \varnothing$.

Corollary 5.1 Let (U, R_p) be an a pre-approximation space, $X \subseteq U$. Then: $\underline{R}_P(X)$ and upper $\overline{R}_P(X)$ are definable (exact) sets.

Example 5.2 In an example 2.1; $X = \{a, c.d\}$ is a pre- exact , $Y = \{a, b\}$ is a pre- rough set .

Proposition 5.3 Every indiscrete approximation space is a quasi-discrete pre-approximation space.

Proof: Obvious.

Proposition 5.4 Let (U, R_p) be a pre-approximation space, then:

(i) Every exact set in U is pre-exact.

(ii) Every pre-rough set in U is rough.

Proof: Obvious.

From Proposition 5.4 and Corollary 5.1, we have:

Corollary 5.2 Let (U, R_p) be a pre-approximation space, $X \subseteq U$. Then: $\underline{R}_P(X)$ and $\overline{R}_P(X)$ are pre-definable (pre-exact) sets.

Remark 5.6 In above proposition, the converse need not be true as it can be seen by the following example.

Example 5.3 In an example 2.1, if $X=\{a,c,d\}$, $\underline{R}(X)=\{a\}$ and $\overline{R}(X)=U$ then X is a rough set but $\underline{R}_p(X)=\{a,c,d\}=\overline{R}_p(X)$ which means that X is not a pre- rough, (X is a pre- exact).

Example 5.4 In an indiscrete space U, for every $X\subset U$, we have $\mathrm{cl}(X)=X$ and $\mathrm{int}(X)=\varnothing$ which means that X is a rough set , but $p.\mathrm{int}(X)=X=p.\mathrm{cl}(X)$, then X is not a pre- rough set (X is a pre-exact)

Definition 5.10: Let (U,R_P) be a pre-approximation space, $X\subseteq U$. Then the pre-positive region of X is denoted by $POS_P(X)$ and defined by $POS_P(X)=\underline{R}_P(X)$. Also the pre-negative region of X is denoted by $NEG_P(X)$, and defined by $NEG_P(X)=U-\overline{R}_P(X)$.

Proposition 5.5 Let (U,R_P) be a pre-approximation space, $X\subseteq U$. Then the following statements are true:

1-$P.B\ (X)\subseteq B(X)$.

2- $NEG\ (X)\subseteq NEG_P(X)$.

Proof: 1- $P.B\ (X)=\overline{R}_P(X)-\underline{R}_P(X)\subseteq\overline{R}_P(X)-\underline{R}(X)\subseteq\overline{R}(X)-\underline{R}(X)=\mathrm{B}\ (X)$.

2- Since $\overline{R}_P((X)\subseteq\overline{R}\ (X)\Longrightarrow U-\overline{R}\ (X)\subseteq U-\overline{R}_P\ (X)$ and
$NEG\ (X)=U-\overline{R}(X)\subseteq U-\overline{R}_P\ (X)=NEG_P\ (X)$, then
$NEG\ (X)\subseteq NEG_P\ (X)$.

Remark 5.7 We can get the pre-approximation operators by following the steps in remark 4.2, after that is found the pre-open sets from the topology τ to get the family $P_O(U,\tau)$ as follows:

Example 5.5 Let $U=\{a,b,c,d\}$, and R be a general relation where:

R = {(a, a), (a, c), (c, b), (c, c), (c, d), (d, b), (d, d)}, then S={{a,c},{b,c,d},{b,d}}, τ ={U,Φ,{a,c},{b,c,d},{b,d},{c}}. $P_O(U,\tau)$ ={U,Φ,{b,d},{b,c},{a,c},{b},{c},{d},{c,d},{a,b,c}, {b,c,d}},
$P_C(U,\tau)$=
{U,Φ,{a,c},{a,d},{b,d},{a,c,d},{a,b,d},{a,b,c},{a,b},{d},{a}}.
If X={a,c,d} then $\underline{R}$ (X)={a,c}⇒ $\underline{R}_P(X)$)={a,c,d}, $\bar{R}$ (X) =U ⟹ $\bar{R}_P(X)$=U. Also, $P.B(X) = \{b\} \subseteq\ B(X) = \{b,d\}$.

Definition 5.11 Let (U,R_P) be a pre-approximation space, X⊆U Then the pre accuracy of X is denoted by $\eta_p(X)$ and defined by:

$\eta_p(X)$ =$\frac{|\underline{R}_P(X)|}{|\bar{R}_P(\mathrm{X})\,|}$, where | $\bar{R}_P(X)$ |≠Φ.

Proposition 5.6 Let (U,R_P) be a pre-approximation space, X⊆U. Then: $(X) \leq\ \eta_p(X)$.

Proof: Obvious.

Example 5.6 Using the same general relation as in (Example 5.4)
If X= {a, c, d}, then $\eta(X) = \frac{|\underline{R}\ (\mathrm{X})|}{|\bar{R}\ (\mathrm{X})|} = \frac{1}{2}$ and $\eta_p(X) = \frac{|\underline{R}_P(X)|}{|\bar{R}_P(\mathrm{X})\,|} = \frac{3}{4}$.
Thus $(X) \leq\ \eta_p(X)$.

Original rough membership function is defined using equivalence classes. It was extended to topological spaces [3,7,8,11,12], namely, If τ is a topology on a finite set X⊆U, where its base is β, then the rough membership function is $\mu_X^P(x) = \frac{|\{\cap \mathrm{Bx}\}\cap \mathrm{X}|}{|\{\cap \mathrm{Bx}\}|}$, x∈X, where Bx is any member of β containing x .

We introduce the following definition for a pre-rough membership function to express $P.B(X), POS_P$ (X), NEG_P (X), for a subset X⊆U.

Definition 5.12 Let (U,R_P) be a pre-approximation space, X⊆U. Then the pre-rough membership function on U is:

$\mu_X^P(x)$:⟶ [0,1], and it is defined by

$$\mu_X^P(x) = \begin{cases} 1, & \text{if } 1 \in B_x(X) \\ \min B_x(X, & \text{otherwise'} \end{cases} \quad where$$

$B_x(X) = \{\frac{|B \cap X|}{|B|} : B \text{ is a Pre-open set}, x \in B\}$.

Definition 5.13 The pre-rough membership function may be used to define pre-approximations and pre-boundary region of a set, as shown below:

$\underline{R}_P(X)$ ={ $\{x \in U: \mu_X^P(x))$=1}, $\overline{R}_P(X)$={$x \in U$: : $\mu_X^P(x))$ >0},

$P.B(X)$ ={$x \in U$: 0 < $\mu_X^P(x)$ <1}.

Definition 5.14 Let (U, R_P) be a pre-approximation space, $X \subseteq U$. Then we have:-

$POS_P(X)$ = {$x \in U$: $\mu_X^P(x)$ =1}.

$NEG_P(X)$ = {$x \in U$: $\mu_X^P(x)$=0}.

$P.B(X)$= {$x \in U$: $\mu_X^P(x)$<1}.

Example 5.7 Let U={a,b,c,d } , R={(c,c),(b,b),(b,d)}, S={{c},{b,d}}, So τ ={U,Φ,{c},{b,d},{b,c,d}}. If X= {a,b,c}, then we get $\mu_X^P(a)$=1, $\mu_X^P(b)$ = 1, $\mu_X^P(c)$ = 1, $\mu_X^P(d)$ = 0.

From pre-membership function, we get $\underline{R}_P(X)$={a,b,c},

$\overline{R}_P(X)$={a,b,c}, $P.b(X)$ =Φ, $POS_P(X)$ ={a,b,c}, $NEG_P(X)$ ={d}.

Thus, X is a Pre-$definable$ (Pre-$exact$) set.

Proposition 5.8 In every pre-approximation space (U, R_p) the following hold:

(i) The intersection (union) of two exact sets is exact.

(ii) The complement of an exact set is an exact.

(iii)The difference between two exact sets is an exact

Proof: Using the properties of clopen sets, the proof is obvious.

Proposition 5.7 is not satisfied in the case of pre-exact sets. We can see that in the following example.

Example 5.8 In an Example 5.1, the set $X = \{a, c, d\}$ and $Y = \{a, b, d\}$ are pre-exact sets, while $X \cap Y = \{a, d\}$ is not pre-exact. It is pre-rough set.

Definition: 5.15 Let (U, R_p) be a pre-approximation space. Then $X \subseteq U$ is said to be:

(i) Totally pre-definable (exact) set if $X = \underline{R}_P(X) = \overline{R}_P(X)$ $(i.e., P.B(X) = \emptyset)$.

(ii) Internally pre- definable set if $X = \underline{R}_P(X)$, such that $P.\underline{Ed}_R(X) = \emptyset$.

(iii) Externally pre-definable set if $X = \overline{R}_P(X)$, such that $P.\overline{ED}_R(X) = \emptyset$.

(iv) pre-Undefinable or (pre-rough) set if $X \neq \underline{R}_P(X) \neq \overline{R}_P(X)$ $(i.e., P.b(X) \neq \emptyset)$

6. Conclusion:

In this paper, we initiated a topological structure based on a binary relation to classification of sets in generalization of approximation spaces that open the way to more topological applications in rough set context and help in formalizing many application. In addition, using pre-approximation space, we can obtain (12) dissimilar granules of the universe of discourse. Our approach is the largest granulation based on pre-open sets in topological spaces. Some important concepts and properties of Pawlak rough sets and rough membership function are generalized, we have seen how the topology represent the magic box for definability of sets and the golden tools to measure exactness or the roughness of sets

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