Linear Fractional Distributed Order Based Derivative and Entropy

Dr.Eldakli Mohsan[1] , Freshk Marwa[1]
University of Zawia, Faculty of Science, Physics department
Az Zawiyah, Libya.
E-mail address: m.amara@zu.edu.ly

الملخص:

اشتقاق دالتين جديدتين للإنتروبي الجزئية، الأولى عبارة عن امتداد لدراسة اوبرياكو وشافي والثانية تعميم من خلال استخدام أوسع لمشتقة التوزيع الخطي الجزئي.

الإنتروبي الأولى هي دوال ذات متغيرين، والثانية مزيج من الدوال الخطية التي لهم نفس خصائص إنتروبي شانون باستثناء الجمع.

فان كلاهما يحقق القانون الثالث للديناميكا الحرارية وفقاً فعندما $q_{1,}\ q_2 \in (0,1]$ لدراسة بينتو ، وعندما $q_1 \geq 1$ و $q_2 \leq 1$ تتحقق معايير الاستقرار لليتشي.

Abstract:
Two new fractional entropy functions, first, based on extension of Ubriaco and Shafee approach, and second, the generalization through the use of concept of expansion of linear fractional distributed order derivative, is proposed. The first entropies are the two-parametric functions. The second entropy is a linear combination of the above functions. Then they have the same properties as the Shannon entropy except additivity. For $q_{1,}\ q_2 \in (0,1]$, these entropies satisfy the third law of thermodynamics in the Bento sense, and, for $q_1 \leq 1$ and $q_2 \geq 1$, the Lesche stability criteria.

Keywords: entropy, fractional calculus, Lesche stability, third law of thermodynamics.

1.Introduction:

Statistical entropy is a measure of the number of possibilities or randomness available to a system, and assumes it is minimally zero when the system is in a given state and maximal value when a system can be in a number of microstates randomly with equal probability, with no uncertainty in its description. Thermodynamic entropy, as opposed to the previous, should describe the system as a whole, and not just one of its microscopic parts. Over the past three decades, there has been a lot of interest in generalizing the Shannon entropy and exploring the consequences of applying these new concepts in several scientific fields [1] to [6]. For the new entropy functions are considered properties characteristic for the Shanon entropy: non-negativity, additivity, monotonicity and continuity, extensivity, convexity, stability, and, particularly, whether they conform to the third law of thermodynamics [7] and [8].

As a consequence of the mentioned, central tendency to the development of the statistical mechanics of systems is the definition of the free energy. The existence of this function is the result of normalization of the probability distribution function, which in turn controls the behaviour of all the macroscopic properties of the ensemble. The majority entropy functions depend on an additional parameter q and become the Shannon entropy function when this parameter takes the value $q = 1$.

These generalizations mostly could be non-extensive and opening the possibility for applications to systems with long range interactions between macroscopic parts and non-additivity of energies on macroscopic scales.

The concept of derivative or integral operators is traditionally associated to an integer in terms of the number of applications to the given function. The main idea is to examine the properties of the ordinary derivative and see where and how it is possible to generalize the concepts to the fractional operators.

Precise mathematical formulation of basic fractional calculus (FC) or its many applications are given in [9] and [10]. Several problems in mathematical physics and engineering have been modeled via distributed order fractional calculus (DOFC) [11] to [14].

In FC frequently used left Riemann-Liouville, Caputo or Weyl fractional derivatives. Using them determined appropriate

distributed order fractional operators. They are known expressions for entropy inspired in the properties of basic FC [5] to [7].
Order of derivative operators is a strong connected to parameter *q*. In the statistical mechanics, the main motivation to propose new entropies to be able to describe phenomena that lie outside the scope of the Boltzmann–Gibbs formalism.
In the present paper, we introduced a new entropy function based on generalized linear DOFC. For that porpuse is notice that the Shafee entropy [3] and the Ubriaco entropy [5] can natural generalize into two-parameter concept. After that, considers properties of expansion mentioned two-parameter entropy through their specific linear combinations.
Letter is organized as follows. In section 2, introduced, some necessary definitions and mathematical preliminaries of FC. In section 3, derived the new entropy functions in the spirit of the standard FC. In section 4, described some properties of this entropies. Finally, section 5 outlines the main conclusions.

2. Preliminaries and notations:
In the literature exists various definitions of fractional and distributed order derivatives. One of these definitions of a fractional and distributed order derivative is the Weyl definition. The Riemann-Liouville fractional derivative of order *q* is defined as [9] and [10]

$$\left({}_{RL}D_t^q f\right)(t) := \frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_a^t dt' \cdot \frac{f(t')}{(t-t')^{q+1-n}},\ n-1 < q \le n,\ n \in \mathbf{N}. \qquad (1)$$

If $a = -\infty$, (1) called Weyl's fractional derivative. If Re $(\lambda) > 0$, for Weyl's fractional derivative valid relation:

$${}_W D_t^q e^{\lambda \cdot t} = \lambda^q e^{\lambda \cdot t}. \qquad (2)$$

The distributed order derivative based on the Weyl's derivative[10] and [12], if exists, is:

$$\left({}_{W}\mathbf{D}_{t}f\right)(t) := \left(\int_{0+}^{\infty} dq' \cdot A(q')\, {}_{W}D_{t}^{q'} f\right)(t). \tag{3}$$

Both defined fractional derivative operators are linear. Sometimes, as in [13], used additional conditions, e.g. for (3), $A(q')$ is a non-negative function.

3. Establishing to the new entropy concept:

The Tsallis entropy observation can be defined from the equation [1] and [4] to [15]

$$S = \lim_{t \to -1} D_q^t \sum_i p_i^{-t}, \tag{4}$$

Opened the possibility to define new entropy functions [16]. Where the operator $D^t{}_q$ is called the Jackson q-derivative [17] defined as

$$D_q^t := t^{-1} \frac{1 - q^{t\frac{d}{dt}}}{1 - q}. \tag{5}$$

Parameter q have a real values and sometimes called entropic index, p_i is the probability.
The equation (4) usually written in the form

$$S_q = \frac{1}{1-q} \sum\nolimits_i \left(p_i^q - 1\right)$$

In the limit $q \to 1$ the Shanon entropy recovered. To the best author knowledge, Rénya entropy [2] has not yet been written in the form of some type of differential calculus. The most studied generalization of the Shannon entropy, however the Rényi entropy and the Tsallis entropy.
Ubriaco [5] proposed entropy functions based on FC which has a physical sense [7]

$$S := \lim_{t \to -1} {}_{W}D_t^q \sum_i e^{-t \cdot \ln p_i}, \; 0 < q \le 1. \tag{6}$$

In another form Eq. (6) can be written as follows

$$S=\sum_i p_i\cdot\left(-\ln\left(p_i\right)\right)^q \tag{7}$$

Setting $t\to b$ and, if defined two-parameter fractional derivative operator ${}_W D_b^q f(b):=\lim_{t\to b}{}_W D_t^q f(t)$, then the new and direct generalization of the Eq. (6) is

$$S_{q_1,q_2}[p]:=\left({}_W D_{-q_1}^{q_2}\sum_i e^{q_1\cdot\ln p_i}\right). \tag{8}$$

That the entropy becomes the function

$$S_{q_1,q_2}[p]=\sum_i p_i^{q_1}\cdot\left(-\ln\left(p_i\right)\right)^{q_2}. \tag{9}$$

Where, $S_{q_1,q_2}[p]=\sum_i s_{i,q_1,q_2}[p]=\sum_i \phi_{q_1,q_2}(p_i)$ is a continuous positive non-additive function. If $q_1 = q$ and $q_2 = 1$, then (9) described the Shafee entropy function [3].

One interpretation for his entropy function considered fractional values of cell numbers q (a fractional size of the register) with the probability that the whole "new phase space" is occupied by the given letter - p_i^q. The introduction of fractional values of cell numbers can be taken in the same spirit as defining the fractal (Hausdorff) dimensions of dynamical attractors and in complex systems [18] and [19].

According to construction, in this form, the new entropy function S_{q_1,q_2} represents the final result, within the specified FC and constraint equation. An extension of this function it is possible inside the framework of ideas of DOFC. In this sense, using the definition given by equation (3) and, if exists, the new fractional derivative is given by the equation:

$$\left({}_W\mathbf{D}f\right)(t):=\int_{0+}^{t} dq_1'\int_{0+}^{\infty} dq_2'\cdot A\left(q_1',q_2'\right){}_W D_{-q_1'}^{q_2'} f\left(-q_1'\right). \tag{10}$$

If $t\to\infty$ and substituting $A(q_1',q_2')=\delta(q_1'-t)A(q_2')$ in (10) obtained (3). ($\delta(x)$ is the Dirac delta function), and, for $A(q_1',q_2')=\delta(q_1'-q_1)\delta(q_2'-q_2)$, $q_1 < t$, obtained equation (9).
The new distributed order derivative based on the Weyl's derivative.
The equation (10) is a linear operator. Therefore, for $t\to\infty$, new main entropy is given by the equation:

$$S[p]:=\int_{0+}^{\infty} dq_1'\int_{0+}^{\infty} dq_2'\cdot A(q_1',q_2')_W D_{-q_1'}^{q_2'}\sum_i e^{q_1'\cdot\ln p_i}. \qquad (11)$$

The physical sense of the definition of (11) is that they are describing multifractal or more complex systems.
A (q_1, q_2) is a function or distribution. It will be shown in the next subsection that it can, under some certain conditions to realistically modeling natural processes.

3.1. Some properties one class of the entropy functions:
The model of entropy that is being considered in this section is given by the equation:

$$S[p]:=\int_{0+}^{\infty} dq_1'\int_{0+}^{\infty} dq_2'\cdot A(q_1',q_2')\sum_i p_i^{q_1'}\cdot\left(-\ln\left(p_i\right)\right)^{q_2'}, \qquad (12)$$

where $S[p]$ is a strictly positive function and $A(q_1',q_2')\geq 0$.
All functions shown features of $S[p]$, also are valid in the discrete case:

$$A(q_1',q_2')=\sum_{k=1}^{m_q}\sum_{l=1}^{n_q} A(q_{1k},q_{2l})\delta(q_1'-q_{1k})\delta(q_2'-q_{2l}). \qquad (13)$$

The coefficients in (13) satisfies the inequalit $A(q_{1k},q_{2l})>0$ It is clear that from the condition:

$$\frac{\partial s_{i,q_1,q_2}[p]}{\partial p_i}=0 \qquad (14)$$

$s_{i,q_1,q_2}[p]$ has a maximum at $p_i = -e^{-\frac{q_2}{q_1}}$ with a second derivative at this point given by:

$$\frac{\partial^2 s_{i,q_1,q_2}[p]}{\partial p_i^{\,2}}\Big|_{p_i=-e^{-\frac{q_2}{q_1}}} = -q_2 e^{-\frac{q_2}{q_1}(q_1-2)}\left(\frac{q_2}{q_1}\right)^{q_2-1}. \tag{15}$$

Therefore, function given by (12) has at least one maximum.

The reasons for this are, except continuity, concavity and positivity of $\phi_{q_1,q_2}(p_i)$,are:

(1) minimum $\phi_{q_1,q_2}[p_i \to 0] \to 0$

(2) minimum $\phi_{q_1,q_2}[p_i \to 1] \to 0$

(3) $A(q_1', q_2') \geq 0$.

In particular, the binary entropy:

$$S^{bin}_{q_1,q_2} = p^{q_1}(-\ln p)^{q_2} + (1-p)^{q_1}(-\ln(1-p))^{q_2} \tag{16}$$

It has a maximum at $p = 0.5$.

The same is true for $S[p] = S^{bin}[p]$. As expected, by definition, $S[p]$ is non-additive function. This function is concave, as a non-negative linear combination of concave functions. Based on the statistical-thermodynamic principles, the probability distributions can be obtained by maximizing the corresponding entropy function $S_{q_1,q_2}[p]$ [7] (under the constraints $\Sigma_i\, p_i = 1$ and $\Sigma_i\, p_i\, \varepsilon_i = E$, E is the internal energy of the ensemble per constituent, ε_i is the i-th state energy), subject to constraint equation:

$$L_{q_1,q_2} = S_{q_1,q_2}[p] + \alpha\left(1 - \sum_i p_i\right) + \beta\left(E - \sum_i \varepsilon_i p_i\right) \tag{17}$$

Where α and β are the Lagrange multipliers associated with the normalization of the pdf's p_i and the conservation of energy, such that setting $\frac{\partial L_{q_1,q_2}}{\partial p_j}=0$, leads to the equation:

$$\phi_{q_1,q_2}{}'\left(p_i\right)=\alpha+\beta\varepsilon_i \tag{18}$$

with the solution $\phi_{q_1,q_2}\left(p_i\right)=\left(\alpha+\beta\varepsilon_i\right)p_i$. Hence we get the relation for the pdf

$$p_i=\psi_{q_1,q_2}{}^{-1}\left(\beta\left(\varepsilon_i-A\right)\right) \tag{19}$$

with the definition:

$$\psi_{q_1,q_2}\left(p\right):=\frac{\phi_{q_1,q_2}\left(p\right)}{p} \tag{20}$$

The function A=-α/β is the Helmholtz free energy and the equation for $\phi_{q_1,q_2}{}'\left(p\right)$ is:

$$\phi_{q_1,q_2}{}'\left(p\right)=p^{q_1-1}\left(-\ln p\right)^{q_2-1}\left(-q_1\ln p-q_2\right). \tag{21}$$

For (19) we assume that the function $\psi_{q_1,q_2}\left(p_i\right)$ can be inverted. In the case of the equation (9) for entropy function, the equation (19) is

$$p_i=e^{-\frac{q_2\cdot W\left(\frac{\left(\left(1-q_1\right)\sqrt[q_2]{\alpha+\beta\cdot\varepsilon_i}\right)}{q_2}\right)}{\left(1-q_1\right)}} \tag{22}$$

The function W=$W(z)$ called Lambert function, defined by:

$$W\left(z\right)e^{W(z)}:=z. \tag{23}$$

If q_2=1 and q_1=q, the equation (22) replace the appropriate relation in [3].

Solving the similar problem given by (12) is much more complicated and will be subject to the following papers. The study of the stability properties of entropy functions is one of the important issues that need to pay attention to many works. In the framework of the above, Lesche, in a pioneering Articles [20] and [21], proposed a criterion to study the stability of the Rénya entropy function [2].

$$S_q = \frac{\ln\left(\sum_i p_i^q\right)}{1-q} \tag{24}$$

Lesche's main result is that the Rénya entropy is unstable for every value of the q parameter with the exception of 1 (Shanon entropy). The basic motive for existence of this type of stability is to check whether existence of quantitative sensitivity to changes when the probability assignments p on a set of n microstates is perturbed by an infinitesimal amount δp. To some generalizations of the Shannon entropy, these criteria have already been applied, [22] and [24]. Authors in the [23] derived a simple condition from which Lesche stability can be addressed. Let p and p' be two probability assignments. Can be shown that Lesche stability is satisfied if :

$$\frac{\left|S_{q_1,q_2}(p') - S_{q_1,q_2}(p)\right|}{S_{q_1,q_2}^{\max}} < C\sum_{j=1}^{n}\left|p_j - p'_j\right|, \tag{25}$$

Where the constant C is given by:

$$C = \frac{\phi'(0+) - \phi'(1-)}{\phi'(0+) - \int_0^1 dt \cdot \phi'(t)}, \tag{26}$$

Correct values for constant C are real numbers without zero. If $q_1 \leq 1$ and $q_2 \geq 1$ for (26) valid limit:

$$C = \lim_{t \to 0+} \frac{\phi'(t)}{\phi'(t) - \frac{\Gamma(1+q_2) - q_2\Gamma(q_2)}{q_1^{q_2}}} = 1. \tag{27}$$

Still, it should be mentioned the following.
In [25] provide a counterexample to show that the generic form of entropy $S[p] = \sum_i \phi(p_i)$ is not always stable against small variation of probability distribution - Lesche stability even if ϕ is concave function on [0,1] and analytic on (0,1].
Nevertheless, entropic function given by the (9) in certain cases satisfies the third law of thermodynamics, which is demonstrated in the similar as in [7] and [8]. Since it should be satisfied for any suitable entropy expression independent of the Hamiltonian, the third law of thermodynamics has been introduced as a test for the generalized entropies [8].
In the statement [8] is to express this law in the terms of micro-probabilities by assuming that the physical system has ordered microscopic energies ε_i where i = 0, 1,.., N, with no degeneracy. Then, β_i is the contribution of the I-th energy level to the inverse temperature β

$$\beta_i = \frac{\partial S}{\partial p_i}\left(\frac{\partial E}{\partial p_i}\right)^{-1}, \tag{28}$$

Where $\beta = \Sigma_i \beta_i$. As suggested in [8], the third law dictates certain divergence of temperature, occurs if and only if when the entropy vanishes. When $\{p_j\} \to 0$, $j > 0$, as $p_0 = 1$ showing that only the ground state is occupied while all the other states are not ($p_0 \approx 1-p_j$). This case is one check for the third law. Similar to the statement [7], using the equation (9), expression for first derivative by p_j of $S_{q1,q2}$ i :

$$\frac{\partial S_{q_1,q_2}}{\partial p_j} = p_j^{q_1-1}\left(-\ln p_j\right)^{q_2-1}\left(-q_1 \ln p_j - q_2\right) - p_0^{q_1-1}\left(-\ln p_0\right)^{q_2-1}\left(-q_1 \ln p_0 - q_2\right) \tag{29}$$

Which is in the form of (29). Nevertheless, the system energy given by:

$$\frac{\partial E}{\partial p_j} = \varepsilon_j - \varepsilon_0. \tag{30}$$

The final relation for β_j is:

$$\beta_j = \frac{p_j^{q_1-1}\left(-\ln p_j\right)^{q_2-1}\left(-q_1 \ln p_j - q_2\right)}{\varepsilon_j - \varepsilon_0} - \frac{p_0^{q_1-1}\left(-\ln p_0\right)^{q_2-1}\left(-q_1 \ln p_0 - q_2\right)}{\varepsilon_j - \varepsilon_0}. \tag{31}$$

Having substituted $p_0 = 1$, is obtained by:

$$\lim_{p_j \to 0} \beta_j = \lim_{p_j \to 0} \frac{p_j^{q_1-1}\left(-\ln p_j\right)^{q_2-1}\left(-q_1 \ln p_j - q_2\right)}{\varepsilon_j - \varepsilon_0} + \infty = \infty \tag{32}$$

For q_1, q_2 $\in$ (0,1], taking into account the equation (32), in the form "$\infty + \infty$", as in [13].
Also holds, if $q_1 > 1$, then equation (32) diverges to ∞, but, in the form "0 + ∞". For the Lesche stability limitations is the characteristic form "∞ + 0". Author required first result on the right side of the equation (32) as the correct: q_1, q_2 $\in$ (0,1]. Exists, however, some problems. For instance, Bento et al in [7], by their discussion, are not explicitly considered systems that have long-range interactions, and this represents a justifiable limitation of the previous review. In conclusion, constraints for parameters q_1 and q_2, derived from the third law of thermodynamics, the Lesche stability or analogous, may corrected boundary values of the integration in the equation (12).

5. Conclusions:

This paper presented a generalizations of the concept of entropy inspired in the properties of (FC). Within context of the new (FC), defined a two new entropy functions. These new entropies are concave, positive definite, non-additive, for given set of values of two parameters satisfies Lesche stability and the third law of thermodynamics. **Probability distributions based on the** Lambert function **obtained by maximizing the first entropy function together with** the appropriate **constraint equation. The method presents in this study can be used for direct generalization of the Tsallis entropy -equation (4). However, the Tsallis entropy easily** described using the equation (12). According to the author, it is possible to assume that over a can express the entropy for which there are for now no fractional derivatives, as the Rénya entropy - equation (24). Therefore, can be appropriate to ask the following question: whether the equation (12) or similar, a generally enough to include all physically acceptable, the current entropic functions?

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