Approximations of Ideals in Local rings

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Abstract
We consider the rough ideals of a local ring by using the concepts of rough set. We can introduce the rough ideal in the local ring as the generalization of the concepts of Ideal in local ring. Moreover, we will study the rough ideals of a local ring. However, the properties of theses approximations of ideal in local rings are studies.
Keywords: local ring, lower approximation, upper approximation, ideal, rough set.

الملخص
ندرس المثالي الخشن في الحلقة المحلية باستخدام مفاهيم المجموعة الخشنة. حيث قدمنا مفهوم المثالي الخشن كتعميم وتوسيع مفهوم المثالي في الحلقة المحلية. علاوة على ذلك، سوف ندرس المثاليات الخشنة للحلقة المحلية. وكلك فإن خصائص هذه التقريبات المثالية في الحلقات المحلية هي تمت دراستها.

1-Introduction

Rough set theory is a new mathematical tool shown by Pawlak [1] in 1982. Rough set theory is based on a concepts upper and lower approximation. The upper approximation is a set of the union of all the equivalence classes, which are subsets of the set, and the lower approximation is the union of all the equivalence classes, which are intersection with set non-empty. Many researchers study and develop this theory and use it in many areas. For example, Biswas and Nanda [2] introduced the notion of rough subgroups and B.Davvaz[3] present the notation of rough subring with respect ideal. The concepts of rough ideal in a semi-group had introduced by Kuroki in [4]. Some concept lattice in Rough set theory has studied by Y.Y. Yao[5] and B.Davvaz [6] introduced the rough prime and rough primary ideals in commutative rings. The author [7], present the concepts of approximations of maximal and principal ideal. In this paper, we introduce the concept of rough ideal of a local ring. In addition, we study the notion of rough prime ideal in a local ring. However, our result will introduce the rough local ring as an extended notion of a classic local ring and we study some properties of the lower and the upper approximations a local ring.

2- Preliminaries

Suppose that U is a nonempty finite set we called universal and ~ an equivalence relation on U. We say that $U/\sim$ is the family of all equivalent classes of ~ and $[x]_\sim$ is an equivalence class in ~ containing an element $x \in U$. We define X^c as the complementation of X in U for any $X \subseteq U$.

Definition 2.1: Suppose that U universal and ~, an equivalence relation on U. We say $\overline{\sim X} = \{x \in U: [x]_\sim \cap X \neq \emptyset\}$ is ~-upper approximation of a set X with respect to ~ , $\underline{\sim X} = \{x \in U: [x]_\sim \subseteq X\}$. ~-lower approximation of a set X with respect to ~, and the boundary region by $BX_\sim = \overline{\sim X} - \underline{\sim X}$. If $BX_\sim = \emptyset$, we say X is exact (crisp) set anf if $BX_\sim \neq \emptyset$, we say X

The maximal idea. Then we will study the upper and lower approximations ideal. We suppose we have a ring $\mathcal{R}$ and I be an Ideal of a ring $\mathcal{R}$, and X be a non-empty subset of $\mathcal{R}$.

Definition 2.2: Suppose that I is an Ideal of a ring R, we called a is *congruent* of b mod I and we write $a \equiv b(mod\ I) if\ a - b \ \in I(1)$

Remark 2-1: The relation (1) is an equivalents relation.

Definition 2.3: Suppose that the universal set U equal the ring R, we defined the upper and lower approximation of X with respect of I as: $\overline{I(X)} = \cup \{ x \in R : (x + I) \cap X \neq \emptyset\}$, $\underline{I(X)} = \cup \{ x \in R : x + I \subseteq X\}$, respectively. And, $BX = \overline{I(X)} - \underline{I(X)}$ the boundary of X with respect of I.

Definition2.4: We say the ideal M in a ring R is a maximal if $M \neq R$ and the only ideal strictly containing M is R.

Remark 2-2: If R is a commutative ring with identity. Then every maximal ideal of R is prime.

Definition 2-5: A *local ring* is a ring with exactly one maximal ideal M_R we denoted by *Loc(R)*.

Example 2-1. Suppose that Z the ring of integers. If *p* a prime number, then *p*Z is a maximal ideal. So, Z is not a local ring.

Example 2-2: Let *F* be a field, then F is a local ring with unique maximal ideal (0).

Example 2.3: Suppose Z/(p) the integers localized at the prime ideal (p). Suppose that R^p is the set of all equivalence classes of fractions a/b where $a \in R$ *and* $b \in S$ where *S* is the complement of *p*. When *R* is an integral domain, R^p is contained in the quotient field QR. So, it $Z_{(p)} = \{a/b \in \mathbb{Q} : p \nmid b\}$ is a local ring with unique maximal ideal $M = \{a/b \in \mathbb{Q} : p|a, p \nmid b\}$.

Definition 2.6: The spectrum of a ring *R* is the set of all prime ideals in R and write *Spec*(*R*), and the set of its maximal ideals is the maximal spectrum of *R*, denoted by *Specm*(*R*);

Note that, $Specm(R) \subseteq Spec(R)$.

The following theorem gives us an alternative criterion for when a ring is a local ring.

Proposition 2-1: Let *R* be a ring. Then *R* is a local ring with maximal ideal *M* if and only if $M = R \setminus R^{\times}$ is an ideal in R.

Proof: ($\Rightarrow$) Suppose that *I* is an ideal of *R*. Let *a* unit $\in I$. We have $I = (R)$.

If *R* is a local ring with maximal ideal *M*. since *M* is a proper ideal (by definition), *M* contains no units, and so: $M \subseteq R \setminus R^{\times}$, But, M is a maximal ideal, and so $M = R \setminus R^{\times}$.

($\Leftarrow$) Suppose that $M = R \setminus R^{\times}$ is an ideal. So, *M* is maximal because If M_1 larger ideal, then M_1 must contain a unit and will not be proper. And If M_2 any other ideal of *R*, then M_2 must be contained in *M* and so $M = R \setminus R^{\times}$ is unique. So *R* is a local ring with unique maximal ideal *M*. ■

Definition: 2-7: Suppose that *R*, *S* are *local rings*. We define the local homomorphism of local rings is a ring map $f : Loc(R) \rightarrow$

loc(S) such that *loc(R)* and *loc(S)* are local rings and such that $f(M_R) \subset M_S$.

Example 2-4: A field is a local ring. Any ring map between fields is a local homomorphism of local rings.

3-Congruence In Local Ring

Definition 3.1: Suppose that *Loc(R)* be local ring and ~ is an equivalence relation, then ~ is called a congruence relation if $(a, b) \in \sim \Rightarrow (a+x, b+x), (x+a, x+b), (ax, bx)$, and $(xa, xb) \in \sim \ \forall\, x \in Loc(R)$.

Proposition 3.1: Suppose ~ be a congruence relation on a local ring (*Loc(R)*) If $(a, b) \in \sim$ and $(c, d) \in \sim$, then $(a\,c, b\,d) \in \sim$ and $(ac, bd) \in \sim \ \forall\, a, b, c, d \in Loc(R)$.

Definition 3.2: A congruence relation ~ on *Loc(R)* is called complete if

(i) $[a+b]_\sim = \{ [a'+b']_\sim : a' \in [a]_\sim , b' \in [b]_\sim$ and

(ii) $[ab]_\sim = \{ [a'b']_\sim : a' \in [a]_\sim , b' \in [b]_\sim$ for all $a, b \in Loc(R)$.

Definition 3-3: A ideal *M* of a *Loc(R)* defines an equivalence relation ~ on *Loc(R)* , called the *Faraj relation*, given by I$r \sim, r'$ if and only if there exists elements *a* and a' of I satisfying $r+a=a'+r'$. The relation ~ , is an congruence relation on *Loc(R)*.

We define *Loc(R)*/~ is the set of all equivalence classes of elements of *Loc(R)* under this ~ and the equivalence class of an element *r* of *Loc(R)* is $[r]_\sim$.

Throughout this paper ~ denotes the *Faraj congruence relation* induced by an ideal M_R of a local ring *Loc(R)*.

Definition 3.4: An ideal M_R of *Loc(R)* is called a ~ M_R ideal if $r+a \in I \Rightarrow r \in I \ \forall\, r \in Loc(R)$ and $\forall\, a \in I$.

Proposition 3-2: Suppose that f: $Loc(R) \rightarrow Loc(R)$ is a local ring map between two local rings *R* and *S*. Then the following are equivalent:

1- f is a local ring map;
2- $f(M_R) \subset M_S$, and

Proof. by definition.

4- Rough ideal of Local ring

In this section, we consider the ideal of local ring by using the concepts of rough set. Some properties of the lower approximation ideal of local ring are studied.

Definition 4-1: Suppose that $Loc(R)$ is a local ring with M_R maximal ideal. Let $X \subseteq R$, we define the upper approximations of X with respect of M_R is $\overline{M_R(X)} = \cup \{x \in Loc(R) : (x + M_R) \cap X \neq \emptyset\}$, and lower approximation of X with respect of M_R is $\underline{M_R(X)} = \cup \{ x \in Loc(R) : x + M_R \subseteq X\}$, Moreover, the boundary is $BdX = \overline{M_R(X)} - \underline{M_R(X)}$. If $BdX = \emptyset$, then, X is rough set with respect M_R.

Example 4.1: Let us consider the ring $loc(R) = \mathbb{Z}_4$. The maximal ideal is $M_R = \{0,2\}$. Suppose that is $X = \{1,2,3\}$. For $x \in Loc(R) : x + M_R$, we get $\{1,3\}$ and $\{0,2\}$. The upper approximations of X with respect of M_R: $\{0, 2\} \cup \{1, 3\}.so, \overline{M_R(X)} = \{0,1,2,3\}$ and the lower approximation of X with respect of M_R :$\underline{M_R(X)} = \{1,3\}$. $BdX = \overline{M_R(X)} - \underline{M_R(X)} = \{0,2\}$. Then X is rough set with respect maximal ideal M_R.

We can study the properties of ideal in next proposition:

Proposition 4-1: For every approximation ($Loc(R)$,M) and Every subset $X \subseteq Loc(R)$ we have:

1) $\underline{M_R(X)} \subseteq X \subseteq \overline{M_R(X)}$;
2) $\underline{M_R(\emptyset)} = \emptyset = \overline{M_R(\emptyset)}$;

Proof:

1) If $x \in \underline{M_R(X)}$, then $x \in \underline{M_R(X)} = \{ x \in Loc(R) : x + M_R \subseteq X$, then $x \in X$, Hence $\underline{M_R(X)} \subseteq X$, next if $x \in X$, $\overline{M_R(X)} = \{ x \in Loc(R) : (x + M_R) \cap X \neq \emptyset\}$,, then $x \in \overline{M_R(X)}$ then $X \subseteq \overline{M_R(X)}$.
2) it easy to see that

■

Definition 4.2: If A and B are anon-empty subset of $Loc(R)$, we denote AB for the set of all finite sums { $a_1b_1+ a_2\, b_2$,…, a_nb_n :$n\in\mathbb{N}$, $a_i\in A$, $b_i\in B$}.i.e: $AB=\sum_{i=1}^{n}(a_i\,.\,b_i)$, $a_i\in A$, $b_i \in B$.

Example: 4-2: Suppose that $\mathbb{C}\,[[x]]$ whose elements are infinite series $\sum_{i=0}^{\infty} a_i\, x^i$ where multiplications are given by ($\sum_{i=0}^{\infty} a_i\, x^i$) ($\sum_{i=0}^{\infty} b_i\, x^i$) = $\sum_{i=0}^{\infty} c_i\, x^i$such that c_n= $\sum_{i+j=n}^{\infty} a_i b_i$ is local. Its unique maximal ideal consists of all elements, which are not invertible. In other words, it consists of all elements with constant term zero.

Proposition 4-2: Let M_R be maximal Ideal of $Loc(R)$, and A,B are non-empty subset of the a local ring, then

1) $\overline{M_R(A)} + \overline{M_R(B)} = \overline{M_R(A+B)}$;
2) $\overline{M_R(A.B)} = \overline{M_R(A)}.\ \overline{M_R(B)}$;
3) $\underline{M_R(A)} + \underline{M_R(B)} \subseteq \underline{M_R(A+B)}$;
.4) $\underline{M_R(A.B)} \supseteq \underline{M_R(A)}.\underline{M_R(B)}$

Proof: To proof 1) we need to show that $\overline{M_R(A+B)} \subseteq \overline{M_R(A)} + \overline{M_R(B)}$ and $\overline{M_R(A+B)} \supseteq \overline{M_R(A)} + \overline{M_R(B)}$. First, suppose $x\in\overline{M_R(A+B)}$ then $(x+M_R)\cap(A+B)\neq\emptyset$ by definition of upper approximation of $A+B$ with respect M_R. So, there exists $s\in(x+M_R)$ and $y\in A+B$, also, $s=\sum_{i=1}^{n} a_i + b_i$ for some $a_i\in A$, $b_i\in B$. We have, $x\in s+M_R=\sum_{i=1}^{n}(a_i+b_i)+M_R=\sum_{i=1}^{n}(a_i+M_R)+(b_i+M_R)$. Then, there exist $x_i\in(a_i+M_R)$, and $s_i\in(b_i+M_R)$ such that $x=\sum_{i=1}^{n} x_i+s_i$.So, $x_i\in\overline{M_R(A)}$ and $s_i\in\overline{M_R(B)}$. Because, $a_i\in(x_i+M_R)\cap A$ and $b_i\in(s_i+M_R)\cap B$. Therefore, $\overline{M_R(A+B)}\subseteq\overline{M_R(A)}+\overline{M_R(B)}$. Second, suppose $x\in\overline{M_R(A)}+\overline{M_R(B)}$, then $x=\sum_{i=1}^{n} a_i+b_i$for some $x_i\in\overline{M_R(A)}$ And $s_i\in\overline{M_R(B)}$. Hence, $(a_i+M_R)\cap A\neq\emptyset$ and $s_i\in(b_i+M_R)\cap B\neq\emptyset$ for $1\leq i\leq n$. Also, $\sum_{i=1}^{n} x_i+s_i \in A+B\ and\ \sum_{i=1}^{n} x_i+s_i \in \sum_{i=1}^{n}(a_i+b_i)+I.thuse\ (x+M_R)\cap(A+B)\neq\emptyset, that\ mean\ x\in\overline{M_R(A+B)},$ so, $\overline{M_R(A+B)}\supseteq\overline{M_R(A)}+\overline{M_R(B)}$.

2) Similar 1) by using definition of AB.

3) &4) Similar way in(1)&(2) by using definition of lower approximation. ■

Example 3.3. Let consider the a local ring$\mathbb{Z}_4$,M_R={0,2} and A={1,2,3 }, B={0,1},then AB=$\sum_{i=1}^{n}(a_i\,.\,b_i)$, $a_i \in A$, $b_i \in B$.AB= {0,1,2,3}. We get $\overline{M_R(A)} = \{0,1,2,3\}$. And $\overline{M_R(B)} = \{0,1,2,3\}$. Then$\overline{M_R(A.B)} = \{0,1,2,3\}$. However, $\overline{M_R(A)}.\ \overline{M_R(B)}$.= {0,1,2,3}. Then, $\overline{M_R(A.B)} = \overline{M_R(A)}.\ \overline{M_R(B)}$ .Moreover, $\underline{M_R(A)} = \{1,3\}$, $\underline{M_R(B)} = \{\}$, $\underline{M_R(A.B)} = \{0,1,2,3\}$.However, $\underline{M_R(A)}.\underline{M_R(B)} = \{\}$. So, $\underline{M_R(A.B)} \supseteq \underline{M_R(A)}.\underline{M_R(B)}$.

Note that: A ring which contains only a finite number of maximal ideals is called semilocal. Each finite ring is a semilocal ring.so,

CONCLUSION:

We present the rough ideals of a local ring by using the concepts of rough set and study the properties of theses approximations of ideal in local rings. However, The local ring has any applications such as optimization theory, discrete event dynamical systems, automata theory, formal language theory and parallel computing and rough sets also has many applications in the above areas. We certainly hope that our work will be very useful both in the theoretical and application aspect. We also propose to work further on this area to bring out many interesting properties of rough and fuzzy ideal in a local ring.

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